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ASYMPTOTIC SOLUTIONS OF STEADILY SPINNING SHALLOW SHELLS OF REVOLUTION UNDER UNIFORM PRESSURE†

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Abstract—The problem of a steadily spinning shallow elastic shell of revolution under a uniform pressure distribution is investigated by perturbation and asymptotic methods. Accurate numerical solutions are also obtained to confirm the adequacy of perturbation and asymptotic solutions. The limiting case of a flat plate is first solved for the entire range of values of the two load parameters. The results provide a different interpretation of the existing nonlinear membrane solution for the same problem. Except for a narrow range of parameter values, the boundary value problem for the shell case is reduced either to the solution of a sequence of linear problems or to the solution of a previously solved nonlinear problem modified by the solutions of a sequence of linear problems. The analysis for the shell problem under a combined centrifugal and pressure loading shows a complex interplay among the load and geometric parameters. It also enables us to consider a number of previously investigated problems as special cases of our problem and to offer a unified treatment of these problems.

1. STATEMENT OF THE PROBLEM

(a) *Introduction*

This paper investigates the elastostatic behavior of steadily rotating, shallow, elastic shells of revolution under a uniformly distributed pressure.§ The shells are in the form of a hollow frustum bounded by two parallel circular edges in planes normal to the axis of revolution. With different types of edge constraints, these structures, including the limiting case of a flat disc, have many applications in space exploration [1, 2] and in other modern machineries. They will be analyzed in this paper by way of a shallow, thin, elastic shell theory which takes into account the simplifications associated with the inherent axisymmetry of our class of problems [3, 4]. The two-point boundary value problem (BVP) for two coupled second order nonlinear ordinary differential equations (ODE) and the auxiliary formulas for the stress and deformation measures appropriate for the analysis of our class of problems will be summarized in the next section. Except for a narrow range of geometric and load parameter values, an appropriate dimensionless form of this BVP contains one or more small parameters. Perturbation and asymptotic methods [5] are therefore appropriate for the analysis of the BVP. Accurate numerical solutions have also been obtained to confirm the predictions of our asymptotic analyses. To be concrete, the asymptotic and numerical results will be given here for shell frusta with an outer edge free of edge tractions and an inner edge clamped to a rigid hub. Shells with other edge conditions can also be similarly investigated.

The main difficulty in an asymptotic analysis of the BVP for our shell model lies in a proper scaling of the unknown dependent variables to arrive at a dimensionless form of the BVP. The proper scaling varies with the range of load and geometric

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§ Within the framework of a shallow shell theory, there is no distinction between normal and axial loadings.

parameter values, and the dimensionless quantities which play the role of small parameters also change accordingly. The correct scalings in most cases are deduced from limiting situations for which exact or adequate approximate solutions are known (or can be obtained); in other cases, they are inferred by physical reasoning. Once the correct scaling is known, an approximate solution by perturbation and/or asymptotic methods [5] is usually straightforward. In cases where an asymptotic solution cannot be obtained in terms of elementary or special functions, the application of perturbation or asymptotic methods still enables us to deduce some qualitative features of the solution. The order of magnitude of the unknowns, the existence of a layer phenomenon, the width of the boundary or internal layer, and other qualitative solution features contribute significantly to an efficient and accurate numerical solution for the same problem [6–10].

Because of its (relative) mathematical simplicity, the Föppl–Hencky nonlinear membrane theory [11, 12] has been widely used in engineering analyses of sheetlike elastic bodies. Simmonds [1] applied this theory to a limiting case of our problem, namely a steadily spinning disc subject to uniform pressure. The Föppl–Hencky theory seems particularly appropriate for the problem as it gives bounded transverse deflections throughout the flat disc even when the inner hole size shrinks to zero. (In contrast, a quasilinear membrane theory gives an unbounded transverse deflection at the center for this limiting case.) For a sufficiently high pressure load magnitude, the nonlinear theory predicts compressive circumferential stresses in some region(s) of the disc. As an ideal elastic membrane would wrinkle in the presence of compressive stresses, it was concluded in [1] that the axisymmetric nonlinear membrane analysis should not be used for such a high pressure magnitude (relative to the magnitude of the radially directed centrifugal load distribution induced by the steady rotation). Our asymptotic analysis for this problem provides a more positive perspective for the Föppl–Hencky theory. In particular, it will be shown that, for the high pressure range, the nonlinear membrane solution is in fact the leading term outer asymptotic expansion of the exact solution, and it alone provides an accurate approximate solution for the in-plane stretching action throughout the entire annular plate. This and other related results enable us to give a new interpretation of Simmonds' work [1]. The insight on the nonlinear membrane theory gained from our asymptotic analyses is one of the highlights of this paper.

For shells, the undeformed midsurface curvature introduces an additional degree of complexity. Aided by the results for some special cases obtained in [2, 6–9], the shell behavior in different regions of the load-geometric parameter space will be analyzed, again by taking advantage of the presence of one or more parameters after proper scalings. Except for narrow ranges of parameter values, the nonlinear BVP for shells will be reduced either to a sequence of linear problems or to a nonlinear problem already solved herein or elsewhere, corrected by terms determined by linear problems. Accurate numerical solutions of the same BVP confirm and complement the results of our asymptotic analyses. As such, the shallow shell problem is effectively solved.

(b) *Steadily spinning shallow shells of revolution under uniform pressure*

In cylindrical coordinates (r, θ, z), the middle surface of a shallow shell of revolution may be described by $z = z(r)$ with $(\mathrm{d}z/\mathrm{d}r)^2 \ll 1$ (Fig. 1). For shallow shells under axisymmetric external loads so that displacement, strain and stress components are independent of the angular coordinate θ, the elastostatics of the shell undergone only infinitesimal strains is governed by two simultaneous second order differential equations [3, 4]

$$D\left[\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r} - \frac{1}{r^2}\phi\right] - \frac{1}{r}(\xi + \phi)\psi = -V \tag{1.1}$$

$$A\left[\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\psi}{\mathrm{d}r} - \frac{1}{r^2}\psi\right] + \frac{1}{r}\left(\xi + \frac{1}{2}\phi\right)\phi = -A\left[r\frac{\mathrm{d}p_H}{\mathrm{d}r} + (2 + \nu)p_H\right] \tag{1.2}$$

for a stress function ψ and a strain function ϕ with $\xi \equiv \mathrm{d}z/\mathrm{d}r$ being the meridional slope

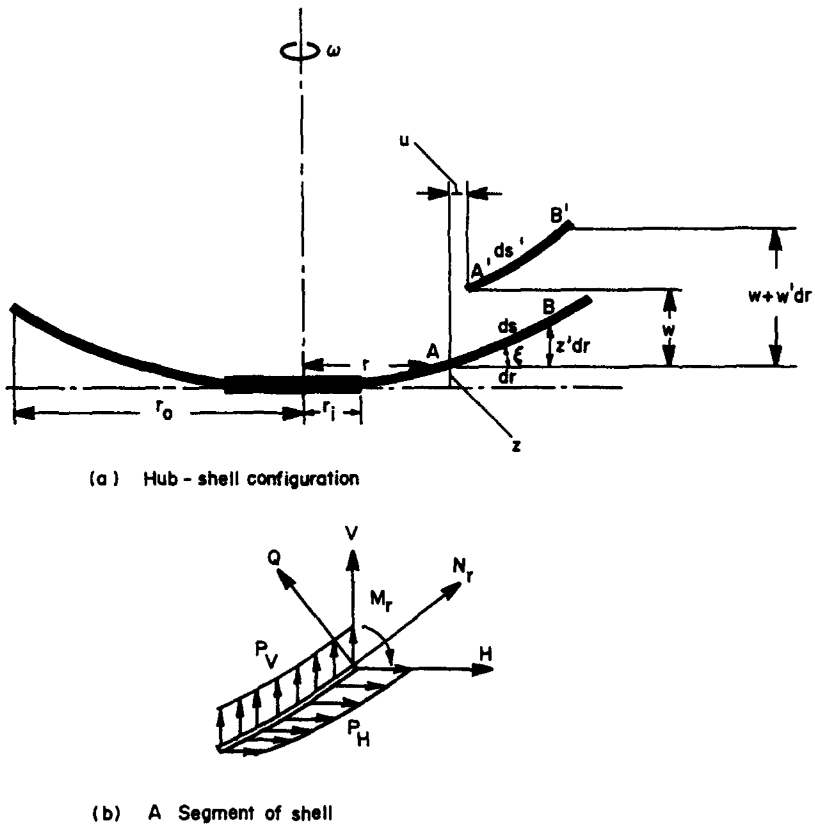

Fig. 1. A normally loaded, spinning shallow shell. (a) Hub-shell configuration. (b) A segment of shell.

angle of the undeformed shell, V being the axial resultant and D and $1/A$ being the bending and stretching stiffness of the homogeneous and isotropic shell of constant thickness h. In terms of Young's modulus and Poisson's ratio, we have

$$D = \frac{Eh^3}{12(1 - \nu^2)}, \quad A = \frac{1}{Eh}. \tag{1.3}$$

The stress resultants and stress couples are calculated from ψ and ϕ by the auxiliary formulas (Fig. 1):

$$N_r = \frac{1}{r}\psi, \quad N_\theta = \frac{d\psi}{dr} + rp_H, \quad Q = V - \frac{1}{r}(\xi + \phi)\psi \tag{1.4}$$

$$M_r = -D\left[\frac{d\phi}{dr} + \frac{\nu}{r}\phi\right], \quad M_\theta = -D\left[\nu\frac{d\phi}{dr} + \frac{1}{r}\phi\right]. \tag{1.5}$$

The midsurface strain and curvature change measures are given by

$$\epsilon_r = A(N_r - \nu N_\theta), \quad \epsilon_\theta = A(N_\theta - \nu N_r) \tag{1.6}$$

$$\kappa_r = -\frac{d\phi}{dr}, \quad \kappa_\theta = -\frac{1}{r}\phi. \tag{1.7}$$

In the absence of transverse shear deformation, axial and radial displacement components w and u (Fig. 1) are given by the strain displacement relations

$$\frac{dw}{dr} = \phi, \qquad u = r\epsilon_\theta. \tag{1.8}$$

Note that $(\xi + \phi) = d(z + w)/dr$ is the meridional slope angle of the deformed middle surface of the shell.

We consider in this paper a shell frustum with $r_i \le r \le r_0$ subject to a uniform normal pressure distribution and to a meridionally varying radial load distribution induced by steadily rotating the shell about its axis of revolution. In this case, we have $p_H = \rho h \omega^2 r$, so that

$$A\left[r\frac{dp_H}{dr} + (2 + \nu)p_H\right] = (3 + \nu)A\rho h\omega^2 r \equiv Ap_0\frac{r}{r_0}, \tag{1.9}$$

and a uniform axial surface load distribution p_V so that

$$-V(r) = -\frac{F_0}{2\pi r} + \frac{1}{r}\int_{r_i}^{r} p_V x \mathrm{d}x = -\frac{F_0}{2\pi r} + \frac{p_V}{2r}(r^2 - r_i^2) \tag{1.10}$$

where F_0 is a constant of integration.

We limit our discussion to the case of a shell with a traction-free outer radial edge $r = r_0$ so that $V(r_0) = N_r(r_0) = M_r(r_0) = 0$. The condition $V(r_0) = 0$ determines F_0 so that

$$-V(r) = -\frac{p_V}{2r}(r_0^2 - r^2). \tag{1.11}$$

The remaining traction free conditions at $r = r_0$, $N_r(r_0) = M_r(r_0) = 0$, may be expressed in terms of ψ and ϕ:

$$r = r_0: \qquad \frac{\psi}{r} = 0, \qquad -D\left(\frac{d\phi}{dr} + \frac{\nu}{r}\phi\right) = 0. \tag{1.12a,b}$$

The inner edge, $r = r_i$, is to be clamped (to a rigid hub) so that

$$r = r_i: \qquad \phi = u = w = 0.$$

The condition of no radial displacement, $u = 0$, may be expressed in terms of ψ by way of $u = r\epsilon_\theta$. Thus the first two clamped edge condition may be written as

$$r = r_i: \qquad \phi = 0, \quad A\left(\frac{d\psi}{dr} + rp_H - \nu\frac{\psi}{r}\right) = 0. \tag{1.13a,b}$$

With (1.9) and (1.11), the two ODEs for ϕ and ψ and the four boundary conditions (1.12) and (1.13) define a two point boundary value problem (BVP) for the determination of ϕ and ψ. All remaining quantities which describe the shell behavior can be obtained from ϕ and ψ and/or their first derivative. The only exception is the axial displacement w which is given by

$$w = \int_{r_i}^{r} \phi(t)\, dt \tag{1.14}$$

where we have made use of the remaining condition $w(r_i) = 0$ for the clamped edge to fix the constant of integration. The present paper is concerned with the characterization of the solution of this BVP for different load magnitudes p_V and p_0.

Edge conditions different from (1.12) and/or (1.13) can also be analyzed. We mention here only the special case of a shell closed at the apex of a dome shaped shell of revolution. For this case, the boundary conditions in (1.13) are replaced by the regularity conditions:

$$r = r_i: \phi = \psi = 0. \qquad (1.15a,b)$$

The second condition along with the boundedness of ψ' implies the symmetry condition of no radial displacement at the apex.

2. FLAT PLATES

(a) *Dimensionless formulation*

Let

$$x = \frac{r}{r_0}, \quad x_i = \frac{r_i}{r_0}, \quad \hat{f}(x) = \frac{\phi}{\bar{\phi}}, \quad \hat{g}(x) = \frac{\psi}{\bar{\psi}} \qquad (2.1)$$

where $\bar{\phi}$ and $\bar{\psi}$ are scale factors to be specified. With $\xi \equiv 0$ for flat plates, the two ODEs for ϕ and ψ, along with (1.9) and (1.11) may be written in terms of the dimensionless quantities as

$$\frac{D\bar{\phi}}{r_0^2}\left(\hat{f}'' + \frac{1}{x}\hat{f}' - \frac{1}{x^2}\hat{f}\right) - \frac{\bar{\phi}\bar{\psi}}{r_0}\frac{1}{x}\hat{f}\hat{g} = -r_0 p_V \frac{(1 - x^2)}{2x} \qquad (2.2)$$

$$\frac{A\bar{\psi}}{r_0^2}\left(\hat{g}'' + \frac{1}{x}\hat{g}' - \frac{1}{x^2}\hat{g}\right) + \frac{\bar{\phi}^2}{r_0}\frac{1}{2x}\hat{f}^2 = p_0 A(-x) \qquad (2.3)$$

where a prime indicates differentiation with respect to x ($[\ ]' = d[\ ]/dx$). To allow for finite deformations, $\bar{\phi}$ and $\bar{\psi}$ must be chosen so that the nonlinear terms are generally not of higher order. This requirement can be met by (tentatively) setting $D\bar{\phi}/r_0^2 = \bar{\phi}\bar{\psi}/r_0$ and $A\bar{\psi}/r_0^2 = \bar{\phi}^2/r_0$, or

$$\bar{\phi} = \frac{\sqrt{DA}}{r_0}, \qquad \bar{\psi} = \frac{D}{r_0}. \qquad (2.4)$$

With (2.4), the two ODEs for $\hat{f}$ and $\hat{g}$ become

$$\hat{f}'' + \frac{1}{x}\hat{f}' - \frac{1}{x^2}\hat{f} - \frac{1}{x}\hat{f}\hat{g} = -k_V \frac{(1 - x^2)}{2x} \qquad (2.5)$$

$$\hat{g}'' + \frac{1}{x}\hat{g}' - \frac{1}{x^2}\hat{g} + \frac{1}{2x}\hat{f}^2 = -k_H x \qquad (2.6)$$

where

$$k_V = \frac{r_0^4 p_V}{D\sqrt{DA}}, \qquad k_H = \frac{r_0^3 p_0}{D}. \qquad (2.7)$$

Correspondingly, the boundary conditions in (1.12) and (1.13) assume the following dimensionless forms:

$$\hat{g}(1) = 0, \quad \hat{f}'(1) + \nu\hat{f}(1) = 0, \qquad (2.8a,b)$$

$$\hat{f}(x_i) = 0, \quad \hat{g}'(x_i) - \frac{\nu}{x_i}\hat{g}(x_i) + \frac{k_H}{3+\nu}x_i^2 = 0. \tag{2.9a,b}$$

The dimensionless load parameters k_V and k_H play an important role in the characterization of the solution of our BVP. If $k_V < 1$ and $k_H < 1$ the above dimensionless form of the BVP is appropriate for a perturbation solution. The special case with $k_V \ll 1$ and $k_H \ll 1$ will be referred to as the *low pressure and slowly spinning case*; a linear theory is expected to be an adequate approximation for this case. Other types of scalings have been worked out for $k_V \gg 1$ and/or $k_H \gg 1$. Briefly, the (k_V, k_H) plane is divided into four regions (see Fig. 2): Region (1.1) with $k_V^2 < k_H < 1$, Region (1.2) with $k_H < k_V^2 < 1$, Region (2) with $k_H > 1$ and $k_V^{2/3} < k_H$, and Region (3) with $k_V > 1$ and $k_H < k_V^{2/3}$. The solution of the BVP exhibits different qualitative features in each of these regions. Accordingly, the method of solution will be different in different regions and will be discussed separately. It is important to emphasize once more that the analyses carried out in this paper are for $x_i \neq 0$ in all cases so that the annular plate has an inner hole of finite size. The discussion on the adequacy of the various linear approximations will have to be modified for the limiting case $x_i \to 0$ (or $x_i \ll 1$).

(b) *A summary of re-scalings for flat plates*

Before we get too involved with the solution process for the BVP, it may be useful to have an overview of the results which are to emerge from our analysis. Evidently, we should expect the solution of our BVP to be qualitatively different depending on the relative magnitude of the parameters k_V and k_H. In the extreme case of $k_V = 0$ and $k_H \ll 1$, we expect the solution to be nearly that of the classical rotating disk problem in linear elasticity theory and the direct stresses to dominate throughout the plate. On the other hand, we expect bending stresses to dominate for $k_H = 0$ and $k_V \ll 1$ and the solution to be nearly that obtained from the classical plate theory. As such, the dimensionless BVP (2.5), (2.6), (2.8) and (2.9) should be re-scaled for specific values of k_H and k_V to more correctly reflect the magnitude of $\hat{f}$ and $\hat{g}$.

To summarize the results for plates (to come) we write the re-scaling of $\hat{f}$ and $\hat{g}$ in the form

$$\hat{f} = c_f f, \qquad \hat{g} = c_g g$$

where the choice of c_f and c_g depends on the value of the parameters k_H and k_V. The re-scaled form of the two ODEs may be written as

$$c_V L[f] - c_{fg}\left[\frac{1}{x} fg\right] = -c_{Vk}\left[\frac{1-x^2}{2x}\right]$$

$$c_H L[g] + c_{ff}\left[\frac{1}{2x} f^2\right] = -c_{Hk}[x]$$

where $L[\ ] \equiv [\ ]'' + x^{-1}[\ ]' - x^{-2}[\ ]$. Of the four boundary conditions, the form of three of them remains invariant upon re-scaling; the fourth may be written as

$$g'(x_i) - \frac{\nu}{x_i} g(x_i) + c_{HB}\left[\frac{x_i^2}{3+\nu}\right] = 0.$$

The nine multiplicative re-scaling factors

$$[c_f, c_g], \qquad \begin{bmatrix} c_V & c_{fg} & c_{Vk} \\ c_H & c_{ff} & c_{Hk} \end{bmatrix}, \qquad [c_{HB}]$$

completely describe the re-scaled problem. The proper re-scalings of the BVP for $\hat{f}$ and $\hat{g}$ in various regions of the k_V, k_H plane which emerge from the analysis of the

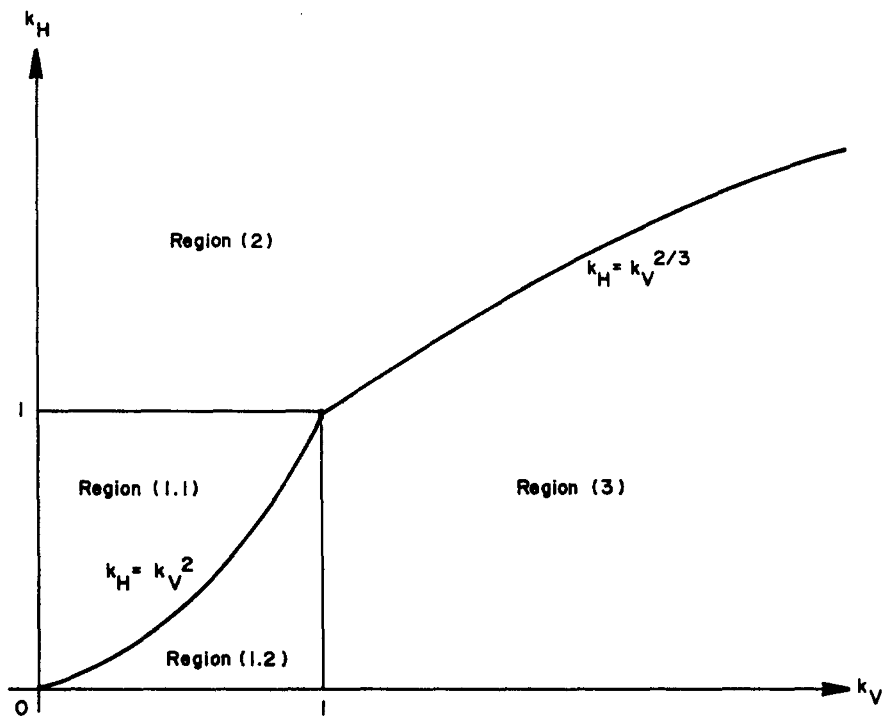

Fig. 2. $k_H - k_V$ parameter plane of flat plates.

next few sections are summarized in Table 1 with the help of these nine multiplicative re-scaling factors.

It is clear from Table 1 that the nonlinear BVP for ϕ and ψ may be reduced to a sequence of linear problems by a perturbation solution in the parameter k_H (or μ_m), k_V^2 and μ_l^3 for the region (1.1), (1.2) and (2), respectively. Except for the range $1 \le k_H \le k_V^{2/3} = O(1)$, a matched asymptotic solution is appropriate for region (3) with the leading term outer solution being the Foppl–Hencky nonlinear membrane solution. Thus, a tractable perturbation or asymptotic solution of the BVP for ϕ and ψ is possible for all combinations of k_H, k_V values except for the narrow region $1 \le k_H \le k_V^{2/3} = O(1)$. The analysis leading to this conclusion and the results summarized above is described in the next three sections.

(c) *Small to moderate load magnitude range*

We consider here the moderate load magnitude range with $k_V < 1$ and $k_H < 1$. For this range of k_V and k_H values, a regular perturbation solution of the BVP is appropriate. Without further restrictions on k_H and/or k_V, two parameter perturbation series in k_H and k_V for $\hat{f}$ and $\hat{g}$ would be necessary and have been worked out in Appendix A of

Table 1. Re-scaling in parameter plane regions for plates

Region in k_H, k_V -plane		Multiplicative re-scaling factors		
(1) $k_V < 1$	(1.1) $\mu_m = \dfrac{k_V^2}{k_H} < 1$	$[k_V, k_H]$,	$\left[\begin{array}{cc|c} 1 & k_H & 1 \\ 1 & \mu_m & 1 \end{array}\right]$,	$[1]$
$k_H < 1$	(1.2) $\mu_m \ge 1$	$[k_V, k_V^2]$,	$\left[\begin{array}{cc|c} 1 & k_V^2 & 1 \\ 1 & 1 & \mu_m^{-1} \end{array}\right]$,	$[\mu_m^{-1}]$
(2) $k_H \ge 1$, $\mu_l^3 = \dfrac{k_V^2}{k_H^3} < 1$		$\left[\dfrac{k_V}{k_H}, k_H\right]$,	$\left[\begin{array}{cc|c} k_H^{-1} & -1 & 1 \\ 1 & \mu_l^3 & 1 \end{array}\right]$,	$[1]$
(3) $k_V \ge 1$, $\mu_l^3 = \dfrac{k_V^2}{k_H^3} \ge 1$		$[k_V^{1/3}, k_V^{2/3}]$,	$\left[\begin{array}{cc|c} k_V^{-2/3} & 1 & 1 \\ 1 & 1 & \mu_l^{-1} \end{array}\right]$,	$[\mu_l^{-1}]$

[13]. To have a better understanding of the qualitative behavior of the solution, we consider separately the two cases $k_H/k_V^2 < 1$ and $k_H/k_V^2 > 1$. A separate treatment of the two cases enables us to use a perturbation series in one parameter only.

Region (1.1): $k_V^2 < k_H < 1$.

For the special case $k_V^2/k_H \ll 1$ (including $k_V = 0$), we expect the solution to approach that of a rotating disk [15]. Therefore, we set

$$\hat{g}(x) = k_H g(x), \qquad \hat{f}(x) = k_V f(x) \tag{2.10}$$

and write (2.5) and (2.6) as

$$f'' + \frac{1}{x}f' - \frac{1}{x^2}f - \frac{k_H}{x}fg = -\frac{1-x^2}{2x} \tag{2.11}$$

$$g'' + \frac{1}{x}g' - \frac{1}{x^2}g + \frac{\mu_m}{2x}f^2 = -x \tag{2.12}$$

where $k_H < 1$ and $\mu_m = k_V^2/k_H < 1$. Correspondingly, the three boundary conditions (2.8a,b) and (2.9a) retain their form with $\hat{f}$ and $\hat{g}$ replaced by f and g while (2.9b) becomes

$$g'(x_i) - \frac{\nu}{x_i}g(x_i) + \frac{x_i^2}{3+\nu} = 0. \tag{2.13}$$

With $\mu_m < 1$ ($k_V^2 < k_H$), we may seek a perturbation series solution in powers of μ_m with k_H as a parameter

$$[f, g] = \sum_{n=0}^{\infty} [f_n(x; k_H), g_n(x; k_H)]\mu_m^n. \tag{2.14}$$

The leading term solutions, g_0 and f_0, are determined by two uncoupled linear BVP

$$\begin{cases} g_0'' + \dfrac{1}{x}g_0' - \dfrac{1}{x^2}g_0 = -x \qquad (x_i < x < 1) \\ g_0(1) = g_0'(x_i) - \dfrac{\nu}{x_i}g_0(x_i) + \dfrac{x_i^2}{3+\nu} = 0 \end{cases} \tag{2.15}$$

$$\begin{cases} f_0'' + \dfrac{1}{x}f_0' - \dfrac{1}{x^2}f_0 - \dfrac{k_H}{x}f_0 g_0 = -\dfrac{1-x^2}{2x} \quad (x_i < x < 1) \\ f_0(x_i) = f_0'(1) + \nu f_0(1) = 0. \end{cases} \tag{2.16}$$

We first solve (2.15) for g_0 and then use the result in (2.16) to get a second order linear ODE for f_0. The BVP (2.15) is identical to the well-known problem of a rotating disk free at the outer edge and clamped to a rigid hub at the inner edge.† If $k_H \ll 1$, the contribution of the $f_0 g_0$ term in (2.16) is negligible. Thus, to leading order, the bending and stretching actions of the plate are completely uncoupled for $\mu_m = k_V^2/k_H \ll 1$ and $k_H \ll 1$.

For moderately small values of μ_m (and k_H not small compared to unity), higher order correction terms in (2.14) will have to be calculated. It is not difficult to see that these higher order terms are also determined (sequentially) by linear boundary value problems and may be obtained by straightforward calculations.

† The solution of this problem is given later in (2.33).

If $k_H \ll (\mu_m <) 1$, we may take instead of (2.14) a perturbation solution in powers of k_H (keeping μ_m as a parameter):

$$[f, g] = \sum_{n=0}^{\infty} [F_n(x; \mu_m), G_n(x; \mu_m)]k_H^n. \tag{2.17}$$

The leading term solutions F_0 and G_0 in that case are also determined by two uncoupled linear BVP in the interval $(x_i < x < 1)$

$$\begin{cases} F_0'' + \frac{1}{x}F_0' - \frac{1}{x^2}F_0 = -\frac{1-x^2}{2x} \\ F_0(x_i) = F_0'(1) + \nu F_0(1) = 0 \end{cases} \tag{2.18}$$

$$\begin{cases} G_0'' + \frac{1}{x}G_0' - \frac{1}{x^2}G_0 + \frac{\mu_m}{2x^2}F_0^2 = -x \\ G_0(1) = G_0'(x_i) - \frac{\nu}{x_i}G_0(x_i) + \frac{x_i^2}{3+\nu} = 0. \end{cases} \tag{2.19}$$

We first solve (2.18) for F_0 and then use the result in (2.19) to get a second order linear ODE for G_0. The BVP (2.18) is just the classical problem of axisymmetric bending by uniform pressure of a flat circular plate clamped at the inner edge and free at the outer edge. If, we have (in addition to $k_H \ll 1$) $\mu_m = k_V^2/k_H \ll 1$ also, the contribution of the F_0^2 term in (2.19) is negligible; the bending and stretching plate action uncouple as we pointed out previously.

For moderately small values of k_H (and μ_m not small compared to unity), higher order correction terms in (2.17) will have to be calculated. It is not difficult to see that they are also determined (sequentially) by linear boundary problems and can be obtained by straightforward calculations.

Region (1.2): $k_H < k_V^2 < 1$.

For the special case $k_H \ll k_V^2 < 1$, (including the limiting case $k_H = 0$ and $k_V \neq 0$), we expect the solution to be approximately that for bending by uniform pressure [16]. Therefore we set

$$\hat{f} = k_V f(x), \qquad \hat{g} = k_V^2 g(x) \tag{2.20}$$

and write the ODE (2.5) and (2.6) as

$$f'' + \frac{1}{x}f' - \frac{1}{x^2}f - \frac{k_V^2}{x}fg = -\frac{1-x^2}{2x} \tag{2.21}$$

$$g'' + \frac{1}{x}g' - \frac{1}{x^2}g + \frac{1}{2x}f^2 = -\frac{1}{\mu_m}x \tag{2.22}$$

where $k_V^2 < 1$ and $\mu_m^{-1} = k_H/k_V^2 < 1$. Correspondingly, three boundary conditions (2.8a,b) and (2.9a) retain their form with $\hat{f}$ and $\hat{g}$ replaced by f and g while (2.9b) becomes

$$g'(x_i) - \frac{\nu}{x_i}g(x_i) + \frac{x_i^2}{\mu_m(3+\nu)} = 0. \tag{2.23}$$

With $k_V^2 < 1$, we may seek a perturbation series solution in powers of k_V^2. The leading term solutions, again denoted by f_0 and g_0, are determined by two uncoupled linear

BVPs:

$$\begin{cases} f_0'' + \frac{1}{x} f_0' - \frac{1}{x^2} f_0 = -\frac{1-x^2}{2x}, \\ f_0(x_i) = f_0'(1) + \nu f_0(1) = 0, \end{cases} \tag{2.24}$$

$$\begin{cases} g_0'' + \frac{1}{x} g_0' - \frac{1}{x^2} g_0 = -\frac{1}{2x} f_0^2 - \frac{1}{\mu_m} x \\ g_0(1) = g_0'(x_i) - \frac{\nu}{x_i} g_0(x_i) + \frac{1}{\mu_m} \frac{x_i^2}{(3+\nu)} = 0. \end{cases} \tag{2.25}$$

We first solve (2.24) for f_0 and then use the result in (2.25) to get a linear BVP for g_0. The BVP (2.24) is just the problem for the linear bending by uniform pressure of an annular plate clamped at the inner edge to a rigid hub and free of traction at the outer edge. Whether or not we have $k_H \ll k_V^2$ ($\mu_m^{-1} \ll 1$), the bending and stretching action of the plate are always weakly coupled as (2.25) gives a nontrivial solution even for the limiting case of a non-rotating plate with $k_H = 0$ (so that $\mu_m^{-1} = 0$). This is in contrast to the plate behavior in region (1.1). Note that higher order terms in the perturbation series are also determined by linear BVPs, the solution of which is straightforward. The special case of $k_H = 0$ has been solved in [16] by a power series solution in x with no restriction on k_V.

(d) *The relatively fast spinning range*

In Region (2) where $k_H > 1$ and $k_V^2 < k_H^3$, the plate is expected to experience significant in-plane extension throughout. This suggests that we let

$$\hat{g}(x) = k_H g(x), \qquad \hat{f}(x) = \frac{k_V}{k_H} f(x) \tag{2.26}$$

and write (2.5) and (2.6) as

$$\frac{1}{k_H}\left[f'' + \frac{1}{x} f' - \frac{1}{x^2} f\right] - \frac{1}{x} f g = -\frac{1-x^2}{2x} \tag{2.27}$$

$$g'' + \frac{1}{x} g' - \frac{1}{x^2} g + \frac{\mu_l^3}{2x} f^2 = -x \tag{2.28}$$

where

$$\mu_l^3 = \frac{k_V^2}{k_H^3}. \tag{2.29}$$

Three of the four boundary conditions (2.8a,b) and (2.9a) retain the same form with $\hat{f}$ and $\hat{g}$ replaced by f and g. The fourth (2.9b) becomes identical (in form) to (2.13).

With $\mu_l < 1$, a perturbation solution for f and g in powers of μ_l^3 is appropriate. The leading term solutions f_0 and g_0 satisfy the ODE

$$\frac{1}{k_H}\left[f_0'' + \frac{1}{x} f_0' - \frac{1}{x^2} f_0\right] - \frac{1}{x} f_0 g_0 = -\frac{1-x^2}{2x} \tag{2.30}$$

$$g_0'' + \frac{1}{x} g_0' - \frac{1}{x^2} g_0 = -x \tag{2.31}$$

and the same boundary conditions as the original problem. The simplification of (2.28) for the leading term approximation is significant as g_0 may now be determined separately. With the boundary conditions,

$$g_0(1) = 0, \quad g_0'(x_i) - \frac{\nu}{x_i} g_0(x_i) + \frac{x_i^2}{3 + \nu} = 0, \tag{2.32}$$

we have

$$g_0(x) = A\left(x - \frac{1}{x}\right) + \frac{1}{8}\left(\frac{1}{x} - x^3\right), \quad A = \frac{1 + \nu}{8(3 + \nu)} \frac{(3 + \nu) + (1 - \nu)x_i^4}{(1 + \nu) + (1 - \nu)x_i^2}. \tag{2.33}$$

Upon substituting (2.33) for $g_0(x)$, (2.30) becomes a linear ODE for f_0

$$f_0'' + \frac{1}{x} f_0' - \left[\frac{1 - Ak_H}{x^2} + k_H\left(A + \frac{1}{8}\right) - \frac{k_H}{8} x^2\right] f_0 = - k_H \frac{1 - x^2}{2x}. \tag{2.34}$$

The two boundary conditions for f_0 are

$$f_0(x_i) = 0, \qquad f_0'(1) + \nu f_0(1) = 0. \tag{2.35}$$

The solution for this linear BVP can be obtained by asymptotic or numerical methods (see Appendix).

For moderately small values of μ_l, higher order terms in the perturbation series in μ_l^3 may be necessary for a more accurate solution. These correction terms are also determined sequentially by linear boundary value problems similar to those for the leading term solution. In particular, we have for the coefficients μ_l^{3n},

$$\begin{cases} g_n'' + \frac{1}{x} g_n' - \frac{1}{x^2} g_n = - \frac{1}{2x} \sum_{k=0}^{n-1} f_k f_{n-1-k} \\ g_n(1) = 0, \quad g_n'(x_i) - \frac{\nu}{x_i} g_n(x_i) = 0 \end{cases} \tag{2.36}$$

and

$$\begin{cases} f_n'' + \frac{1}{x} f_n' - \left[\frac{1}{x^2} + \frac{k_H}{x} g_0(x)\right] f_n = \frac{k_H}{x} \sum_{k=0}^{n-1} f_k g_{n-k} \\ f_n(x_i) = 0, \quad f_n'(1) + \nu f_n(1) = 0. \end{cases} \tag{2.37}$$

Thus, the restriction of relatively fast spinning, $\mu_l < 1$ ($k_V^2 < k_H^3$), reduces the nonlinear problem to a sequence of linear problems with the bending and stretching action of the plate weakly coupled through the forcing terms in the governing (linear) differential equations.

For $k_H \gg 1$, an asymptotic solution of the linear BVP for f_0 is given in the Appendix. Except for a particular solution, the same asymptotic solution is also applicable to f_n. Even without an explicit solution, it is clear from the form of the BVP (2.34)–(2.35) that there is a boundary layer adjacent to both edges of the plate. The (dimensionless) layer width near $x = x_i$ is $O(k_H^{-1/2})$ while, because $g_0(1) = 0$, the (dimensionless) layer width near the turning point $x = 1$ is $O(k_H^{-1/3})$. Furthermore, the layer strength is considerably weaker near the outer edge.

(e) *The relatively high pressure range*

In Region (3) where $k_V \geq 1$ and $k_H^3 < k_V^2$, the plate is expected to experience significant bending throughout. This suggests that we let

$$\hat{g}(x) = k_V^{2/3} g(x), \qquad \hat{f}(x) = k_V^{1/3} f(x) \tag{2.38}$$

and write (2.5) and (2.6) as

$$k_{\bar{\nu}}^{-2/3}\left[f'' + \frac{1}{x}f' - \frac{1}{x^2}f\right] - \frac{1}{x}fg = -\frac{1-x^2}{2x} \tag{2.39}$$

$$g'' + \frac{1}{x}g' - \frac{1}{x^2}g + \frac{1}{2x}f^2 = -\frac{x}{\mu_l} \tag{2.40}$$

where μ_l is as given in (2.29). The three boundary conditions (2.8a,b) and (2.9a) retain the same form with $\hat{f}$ and $\hat{g}$ replaced by f and g. The fourth, (2.9b), becomes

$$g'(x_i) - \frac{\nu}{x_i}g(x_i) + \frac{x_i^2}{\mu_l(3+\nu)} = 0. \tag{2.41}$$

With $\mu_l > 1$, a regular perturbation solution in powers of μ_l^{-1} is possible. The leading terms f_0 and g_0 of such the perturbation series are determined by the differential equations and boundary conditions for f and g but with terms multiplied by $1/\mu_l$ replaced by zero. Evidently the BVP for f_0 and g_0 remains a nonlinear problem and an elementary solution is not possible.

On the other hand, for extremely high pressure loading so that $k_{\bar{\nu}}^{2/3} \gg 1$, a solution of the BVP for f and g by the method of matched asymptotic expansions is appropriate for all $\mu_l \geq 1$. The leading term outer (asymptotic expansion) solution, denoted by $F_0(x)$ and $G_0(x)$, is determined by

$$\begin{cases} F_0 = \dfrac{1-x^2}{2G_0}, \quad G_0'' + \dfrac{1}{x}G_0' - \dfrac{1}{x^2}G_0 + \dfrac{(1-x^2)^2}{8xG_0^2} = -\dfrac{x}{\mu_l} \\ G_0(1) = 0, \quad G_0'(x_i) - \dfrac{\nu}{x_i}G_0(x_i) + \dfrac{x_i^2}{\mu_l(3+\nu)} = 0. \end{cases} \tag{2.42}$$

The two equations in (2.42) are just the governing differential equations for the Föppl–Hencky nonlinear membrane theory [11, 12] (in dimensionless form) used in [1]. Methods of solution for these equations have been discussed there as well as in [17–23]. Higher order corrections terms of the outer solution may also be obtained by a regular perturbation series solution in powers of $k_{\bar{\nu}}^{-2/3}$. Evidently, the outer solution can only satisfy two of the four boundary conditions as the governing differential equations for the nth order correction terms G_n and F_n constitute only a second order system. It is therefore necessary to obtain an inner (asymptotic expansion) solution valid in the neighborhood of each edge which satisfies all four boundary conditions on that edge. The actual asymptotic solution to the original problem is then obtained by matching the outer and inner solutions in some overlapping (intermediate) region of validity to form a uniformly valid composite solution.

In general, the outer solution G_0 cannot be obtained in terms of elementary or special functions. The matched asymptotic expansion technique nevertheless provides us with many qualitative features (such as the order of magnitude of $\hat{f}$ and $\hat{g}$ in various regions of the solution domain, the width of the boundary layers adjacent to each edge, etc.) of the exact solution of the original BVP. They are very useful in obtaining numerical solutions for that problem. For the present problem, we can do more. Even without an explicit solution for G_0 and F_0, it is possible to show (see Appendix and [14]) that G_0 is in fact the dominant term in the exact solution for g of our problem throughout the plate. The nonlinear membrane solution thus provides an accurate approximate solution for the stretching action of the plate whenever $k_{\bar{\nu}}^{2/3}$ is large compared to unity. We will comment further on this point in a later section.

(f) *Comparison with numerical solutions*

To verify the results of our asymptotic analysis, the boundary value problem solver COLSYS [24, 25] has been used to generate numerical solutions of the dimensionless two-point BVP defined by (2.5), (2.6), (2.8) and (2.9). For a given set of input values of the parameters

$$\nu = \text{Poisson's ratio}$$

$$x_i = r_i/r_0, \text{ the inner-to-outer hole radius ratio}$$

$$k_V = p_V r_0/D\sqrt{DA}, \text{ the dimensionless pressure load magnitude}$$

$$k_H = p_0 r_0^3/D \equiv (3+\nu)\rho h\omega^2 r_0^4/D, \text{ the dimensionless radial load magnitude,}$$

COLSYS obtains $\hat{f}(x)$ and $\hat{g}(x)$ to a specified accuracy (usually with an absolute or relative error tolerance not larger than 10^{-5}) by the method of spline-collocation at Gaussian points, with repeated mesh redistributions and refinements. An auxiliary computer program calculates the corresponding distributions of stress resultants, stress couples and displacement components from the auxiliary formulas,

$$N_r = \frac{\bar{\psi}}{r_0}\left[\frac{\hat{g}}{x}\right], \quad N_\theta = \frac{\bar{\psi}}{r_0}\left[\hat{g}' + \frac{k_H}{3+\nu}x^2\right]$$

$$M_r = -\frac{D\bar{\phi}}{r_0}\left[\hat{f}' + \frac{\nu}{x}\hat{f}\right], \quad M_\theta = -\frac{D\bar{\phi}}{r_0}\left[\nu\hat{f}' + \frac{1}{x}\hat{f}\right]$$

$$Q = -\frac{\bar{\phi}\bar{\psi}}{r_0}\left[\frac{1}{x}\hat{f}\hat{g} - k_V\frac{1-x^2}{2x}\right], \quad u = \bar{\psi}A\left[x\hat{g}' - \nu\hat{g} + \frac{k_H}{3+\nu}x^3\right]$$

$$w = r_0\bar{\phi}\int_{x_i}^{x}\hat{f}(t)\,dt.$$

Numerical solutions for a number of check cases were first generated to validate the compute code; these include cases examined in [2, 8–10].

The validated computer code has been used to generate numerical solutions for a large number of new cases to confirm the asymptotic results obtained in this paper. A small selection of the numerical solutions is given in [13] and [14] to bring out the qualitative features of the solutions in different regions of the k_H, k_V-parameter plane. Generally, the truncated perturbation/asymptotic solutions are accurate to the order of the terms retained when the relevant small parameter is 1/10 or smaller.

(g) *On nonlinear membrane theory*

Because of its mathematical simplicity, the Föppl–Hencky nonlinear membrane theory has been widely used in engineering analysis of sheetlike elastic bodies. Of interest here is the use of this theory in the analysis of a steadily spinning disk under a uniform normal pressure distribution [1]. With recent advances in both nonlinear theories of plates and shells and in methods for asymptotic solutions of differential equations, we now see that the Föppl–Hencky theory may be considered as the outer asymptotic expansion of the exact solution of the von Karman plate theory, see (2.42). Physically, the Föppl–Hencky theory is an approximation of the von Karman theory neglecting in the latter the effect of the small bending stiffness. In this section, we examine the range of applicability of the Föppl–Hencky theory taking advantage of the asymptotic results obtained in this paper.

It is evident from our asymptotic results that even within the framework of a moderately small rotation theory, a nonlinear membrane theory is appropriate only in the relatively high pressure range, i.e. $k_H^3 \leq k_V^2$, or in the notation of [1]

$$\bar{k} = 16\, k_V^2/k_H^3 \geq 16.$$

It is found in [1] that compressive circumferential stresses appear for $\bar{k} = \bar{k}_{wp} = 0.3154$ in the limiting case of a vanishing small hole ($x_i \to 0$) and for smaller $\bar{k}$ if $x_i > 0$. Now, compressive stresses induce wrinkling in the elastic membrane and are therefore unacceptable for various design purposes. However, these same compressive stresses pose no design problems when the membrane theory which generates them represents only the outer asymptotic expansion of the exact solution for the von Karman plate theory. The sheet does have a finite bending stiffness, however small, which would induce bending stresses whenever necessary to inhibit the occurrence of wrinkles. In particular, significant bending stresses develop in a region adjacent to a clamped edge of the annular plate where the outer solution ceases to be valid. A different type of asymptotic solution, which incorporates the significant effect of the small bending stiffness, is appropriate in the edge zone. Thus, in the framework of a matched asymptotic expansion solution for the plate problem, the results in [1] obtained from Föppl–Hencky nonlinear membrane theory continues to be useful whether or not the resulting membrane stresses are compressive.

For $\bar{k} = 16k_V^2/k_H^3 < 16$ and $k_H \geq 1$ † [so that we are in Region (2)], we know from the analysis in region (2) that, as long as x_i is not small compared to unity, a "quasilinear" plate theory resulting from a regular perturbation solution in powers of (k_V^2/k_H^3) for the von Karman equations is appropriate for our problem. As we found earlier in (2.31) and (2.32), this theory generates membrane stresses by a linear membrane theory. The bending action of the plate is then adequately determined by a quasilinear bending theory (2.34)–(2.35) which incorporates the effect of the membrane solution just obtained. As we pointed out earlier, the situation may be quite different if $x_i \ll 1$. In fact, it was shown in [1] that a nonlinear membrane theory should be used as it leads to finite transverse deflection throughout the sheet while the (quasi) linear membrane theory gives an unbounded transverse deflection at the center of an annular plate with an infinitesimally small hole. Whether a (quasi) linear or nonlinear membrane theory is adequate for the stretching action of the plate for $\bar{k} < 16$, the corresponding inner solutions are essential for an accurate description of the actual bending action, particularly near the plate edges.

3. SHALLOW SHELLS

(a) *Introduction*

When $\xi \neq 0$ so that the shell is not a flat plate, we may again use the dimensionless quantities introduced in (2.1) along with

$$s(x) = \frac{\xi(r)}{\xi_0}, \quad k_G = \frac{\xi_0 r_0}{\sqrt{DA}} = \frac{\xi_0 r_0}{h}\sqrt{12(1-\nu^2)}, \tag{3.1}$$

where ξ_0 is chosen so that $\max|s(x)| = 1$, and tentatively write the ODEs for ϕ and ψ as

$$L[\hat{f}] - \frac{1}{x}[k_G s(x) + \hat{f}]\hat{g} = -k_V\frac{1-x^2}{2x} \tag{3.2}$$

$$L[\hat{g}] + \frac{1}{x}\left[k_G s(x) + \frac{1}{2}\hat{f}\right]\hat{f} = -k_H x \tag{3.3}$$

where k_V and k_H are given by (2.7) and the differential operator L is given by $L[\ \ ] \equiv$

† It is clear that for $k_H < 1$ (and therefore $k_V < 1$ also), a suitable linearized theory suffices as we are in Region (1.1), again for x_i not vanishingly small.

[ ]'' + x^{-1}[ ]' − x^{-2}[ ]. In terms of $\hat{f}$ and $\hat{g}$, the relevant boundary conditions for the shell case have the same form as those for the plate case given by (2.8) and (2.9). With $k_G = 0$, (3.2) and (3.3) reduce to the corresponding plate equations (2.5) and (2.6). The appearance of the $k_G s(x)$ term in (3.2) and (3.3) introduces an unusual degree of complexity which makes the analysis of the solution of our BVP considerably more involved.

Consider for example the extreme case when the shell is subject only to normal pressure loading. We know from the results in [6–9] that "polar dimpling" may occur for some range of k_G values. That is, depending on the relative magnitude of k_G, k_V and k_H, the solution of our BVP at a given location may be either $\hat{f} \cong 0$ or $\hat{f} \simeq -2k_G s(x)$ and that an internal layer phenomenon may exist at a location determined by the load magnitude. The occurrence of polar dimpling depends on the interplay among k_G, k_V and k_H in a complicated way which is not at all obvious from the analyses for the plate case.

To deduce the correct shell behavior corresponding to different combinations of load and geometric parameter values, the (k_H, k_V, k_G) parameter space will be divided into three principal regions (Fig. 3). In the next few sections, we will, as we did for the flat plate case, work out the appropriate re-scalings of the BVP (3.2), (3.3), (2.8) and (2.9) in the various subregions of these principal regions and discuss the behavior of the shell implied by these re-scalings, again under the assumption that x_i is not vanishingly small so that hole centered at the apex is of finite size. For an overview of the final results before the rather involved analysis, a summary of the proper re-scaling and the appropriate solution process for different regions of the parameter space can be found in Section e and Table 2. As expected, the results for the special case with $k_G = 0$ are identical to those given in Table 1.

(b) *The light load intensity range*

We consider in this section the load magnitude range $0 \leq k_V < 1$ and $0 \leq k_H < 1$, designated as light load magnitude range in the flat plate case. In the load-geometric

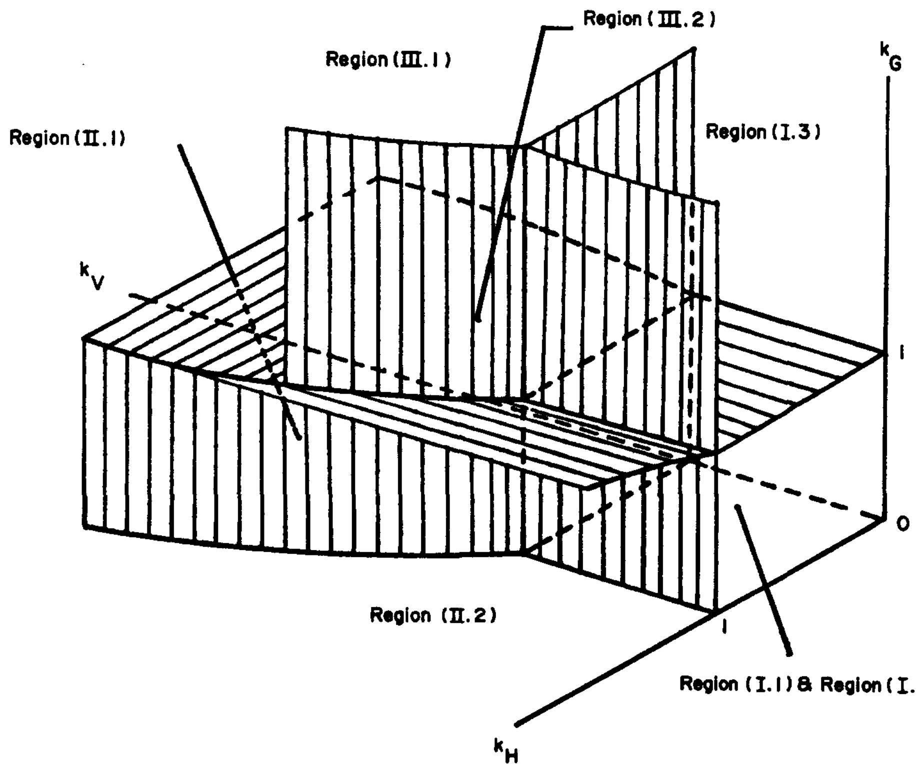

Fig. 3. The (k_V, k_H, k_G) parameter space of shells.

Table 2. Re-scalings in parameter space regions for shells

Region in (k_H, k_V, k_G) space			Multiplicative re-scaling factors		
(I): $k_V < 1$, $k_H < 1$	(I.1): $k_G < 1$, $\mu_G \equiv \frac{k_G}{k_V} < 1$	(I.1.1): $\mu_m \equiv \frac{k_V^2}{k_H} < 1$	$[k_V, k_H]$,	$\begin{bmatrix} 1 & k_H\mu_G & k_H & 1 \\ 1 & \mu_m\mu_G & \mu_m & 1 \end{bmatrix}$,	$[1]$
		(I.1.2): $\mu_m > 1$	$[k_V, k_V^2]$,	$\begin{bmatrix} 1 & k_V^2\mu_G & k_V^2 & 1 \\ 1 & \mu_G & 1 & \mu_m^{-1} \end{bmatrix}$,	$[\mu_m^{-1}]$
	(I.2): $k_G < 1$, $\mu_G > 1$	(I.2.1) $k_H\mu_G > 1$	$[k_H k_G, k_H]$,	$\begin{bmatrix} 1 & 1 & k_H & (\mu_G k_H)^{-1} \\ 1 & k_G^2 & k_G^2 k_H & 1 \end{bmatrix}$,	$[1]$
		(I.2.2): $k_H\mu_G < 1$	$[k_V, \mu_G^{-1}]$,	$\begin{bmatrix} 1 & 1 & \mu_G^{-1} & 1 \\ 1 & k_G^2 & k_V k_G & k_H\mu_G \end{bmatrix}$,	$[k_H\mu_G]$
	(I.3): $k_G > 1$		same as (III); see discussion for $k_H = 0$.		
(II): $k_G < k_V^{1/3}$ or $k_G < k_H^{1/2}$	(II.1): $\mu_l^{-3} \equiv \frac{k_H^3}{k_V^2} < 1$		$[k_V^{1/3}, k_V^{2/3}]$,	$\begin{bmatrix} k_V^{-2/3} & k_G k_V^{-1/3} & 1 & 1 \\ 1 & k_G k_V^{-1/3} & 1 & \mu_l^{-1} \end{bmatrix}$,	$[\mu_l^{-1}]$
	(II.2): $\mu_l^3 < 1$, $k_H > 1$	(II.2.1): $k_H\mu_G < 1$	$\left[\frac{k_V}{k_H}, k_H\right]$,	$\begin{bmatrix} k_H^{-1} & k_H\mu_G & 1 & 1 \\ 1 & k_H\mu_G\mu_l^3 & \mu_l^3 & 1 \end{bmatrix}$,	$[1]$
		(II.2.2): $k_H\mu_G > 1$	$[k_G, k_H]$,	$\begin{bmatrix} k_H^{-1} & 1 & 1 & (k_H\mu_G)^{-1} \\ 1 & k_G^2 k_H^{-1} & k_G^2 k_H^{-1} & 1 \end{bmatrix}$,	$[1]$
(III): $k_V^{1/3} < k_G$, $k_H^{1/2} < k_G$	(III.1): $\mu_l^{-3} < 1$, $k_V > 1$		$[k_G, k_V^{2/3}]$,	$\begin{bmatrix} k_V^{-2/3} & 1 & 1 & k_V^{1/3}k_G^{-1} \\ k_V^{2/3}k_G^{-2} & 1 & 1 & k_H k_G^{-2} \end{bmatrix}$,	$[k_H k_V^{-2/3}]$
	(III.2): $\mu_l^3 < 1$, $k_H > 1$		$[k_G, k_G^2]$,	$\begin{bmatrix} k_G^{-2} & 1 & 1 & k_V k_G^{-3} \\ 1 & 1 & 1 & k_H k_G^{-2} \end{bmatrix}$,	$[k_H k_G^{-2}]$

parameter space this range spans the unit square column with $0 \le k_G < \infty$ and will be designated as Region I of the (k_V, k_H, k_G) space. We subdivide this region into three subregions: Region (I.1) with $k_G < k_V$, Region (I.2) $k_V \le k_G < 1$ and Region (I.3) with $k_G \ge 1$ (see Fig. 4).

Region (I.1): $k_G < k_V$ (and $k_V < 1$, $k_H < 1$).

For $k_G/k_V \ll 1$, we expect the shell to behave very much as a flat plate and the scaling for Region (1.1) or Region (1.2) of the plate problem applies. To be consistent with the flat plate results, we subdivide Region (I.1) for shells into Region (I.1.1) in which $k_V^2 < k_H$ and Region (I.1.2) in which $k_V^2 > k_H$.†

For Region (I.1.1) with $k_V^2 < k_H$, we use the scaling (2.10) and write (3.2) and (3.3) as

$$L[f] - \frac{k_H}{x}(\mu_G s + f)g = -\frac{1 - x^2}{2x} \tag{3.4}$$

$$L[g] + \frac{\mu_m}{x}\left(\mu_G s + \frac{1}{2}f\right)f = -x \tag{3.5}$$

where $\mu_G = k_G/k_V$ and $\mu_m = k_V^2/k_H$ with $\mu_m < 1$ in Region (I.1.1). The boundary conditions for f and g are exactly as in the plate case. With $\mu_G < 1$, a (regular) perturbation solution in (powers of) μ_G is appropriate. The leading term solution is identical

† It is possible to seek a perturbation solution in k_G in Region (I.1) with a leading term identical to the $k_G = 0$ case already treated in Region (1.1) and (1.2) for flat plates. However, for k_G near (but still less than) unity, a different scaling from the one used in Regions (1.1) and (1.2) for flat plates will lead to a more efficient solution. The new scalings in Region (I.2) for shells also provide a gradual transition into Region (I.3).

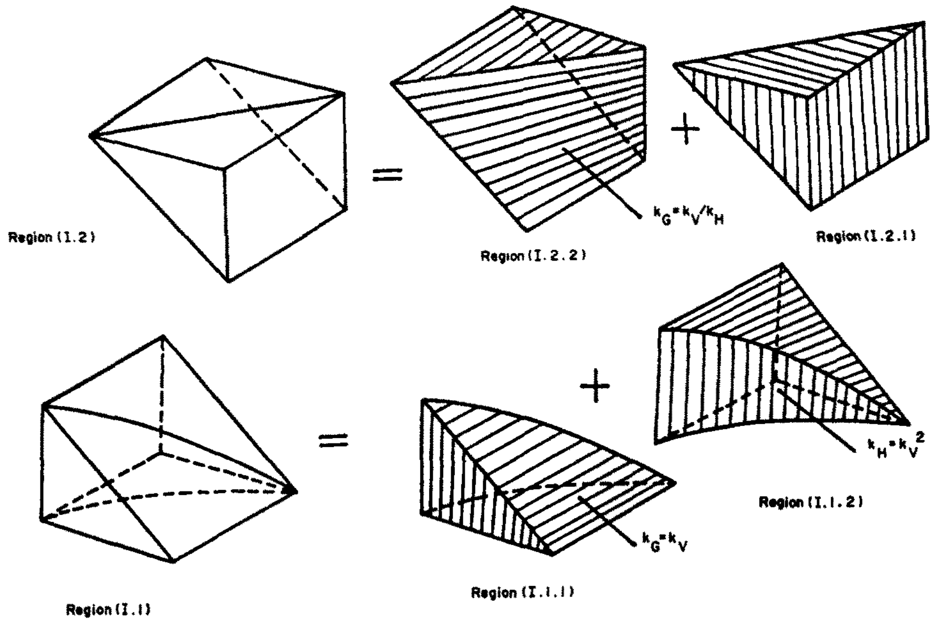

Fig. 4. Region (I) of the parameter space for shells.

to the flat plate solution in Region (1.1); it requires no further comments. Higher order correction terms are determined by a sequence of linear inhomogeneous BVP which differ from each other only in the forcing term.

For Region (1.1.2) where $k_H < k_V^2$, we use the scaling for plate for the same load parameter range given in (2.20) (namely, $\hat{f} = k_V f$ and $\hat{g} = k_V^2 g$). Write the ODE for $\hat{f}$ and $\hat{g}$ as

$$L[f] - \frac{k_V^2}{x}(\mu_G s + f)g = -\frac{1 - x^2}{2x} \tag{3.6}$$

$$L[g] + \frac{1}{x}\left(\mu_G s + \frac{1}{2}f\right)f = -\frac{1}{\mu_m}x \tag{3.7}$$

where $\mu_m = k_V^2/k_H$ is now greater than one. The boundary conditions for f and g in this region are identical in form to those for the flat plate in Region (1.2), namely (2.8a,b), (2.9a) and (2.23). Again a perturbation solution in μ_G (<1) is appropriate. The leading term solution is identical to the flat plate solution in Region (1.2) and requires no further comments. Higher order correction terms in the perturbation series are again determined by a sequence of linear BVP.

Region (1.2): $0 \leq k_V < k_G < 1$ ($k_H < 1$).

With $k_G/k_V \equiv \mu_G > 1$, the scalings used in Region (I.1) are no longer appropriate. As expected, the correct scaling in the new parameter range depends on the relative magnitude of k_V and k_H. Again, we separate two subregions: Region (I.2.1) where $k_G k_H > k_V$ and Region (I.2.2) where $k_G k_H < k_V$.

In Region (I.2.1), the radial load is relatively large compared to the pressure load; the shell response to the combined loading should have a significant component of rotating disc type behavior, especially when $k_G k_H \gg k_V$. Therefore, we set

$$\hat{g} = k_H g, \qquad \hat{f} = k_G k_H f. \tag{3.8}$$

With (3.8), the governing ODE become

$$L[f] - \frac{1}{x}(s + k_H f)g = -\frac{k_V}{k_H k_G}\frac{1 - x^2}{2x}$$
$$L[g] + \frac{k_G^2}{x}\left(s + \frac{1}{2}k_H f\right) f = -x. \tag{3.9}$$

The boundary conditions become

$$f(x_i) = 0, \quad f'(1) + \nu f(1) = 0, \tag{3.10a,b}$$

$$g(1) = 0, \quad g'(x_i) - \frac{\nu}{x_i} g(x_i) + \frac{x_i^2}{3 + \nu} = 0, \tag{3.11a,b}$$

With $k_G < 1$ and $k_V/k_H k_G < 1$, a perturbation solution in k_G^2 is appropriate for the new BVP. The leading term solution $\{f_0, g_0\}$ is determined by

$$L[g_0] = -x \tag{3.12}$$

$$L[f_0] - \frac{k_H}{x} g_0 f_0 = \frac{s}{x} g_0 - \frac{k_V}{k_G k_H}\frac{1 - x^2}{2x}. \tag{3.13}$$

Thus, to a first approximation, the problem is uncoupled into a rotating disc problem and a problem of linear bending of a "pre-stressed" flat plate by a (nonuniform) distribution of pressure load. If $k_H \ll k_G^2 < 1$, a perturbation solution in k_H is also appropriate.

In Region (I.2.2), we have $k_V \geq k_H k_G$ and the pressure load is mainly responsible for the shell behavior. Therefore, we set

$$\hat{f} = k_V f, \qquad \hat{g} = \frac{k_V}{k_G} g \tag{3.14}$$

and write the governing ODE as

$$L[f] - \frac{1}{x}\left(s + \frac{k_V}{k_G} f\right) g = -\frac{1 - x^2}{2x} \tag{3.15}$$

$$L[g] + \frac{k_G^2}{x}\left(s + \frac{1}{2}\frac{k_V}{k_G} f\right) f = -\frac{k_H k_G}{k_V} x. \tag{3.16}$$

Three of the boundary conditions for this range are also given by (3.10) and (3.11a); the fourth becomes

$$g'(x_i) - \frac{\nu}{x_i} g(x_i) + \frac{k_H k_G}{k_V}\frac{x_i^2}{3 + \nu} = 0. \tag{3.17}$$

The form of the new BVP suggests that a perturbation solution in k_G^2 would reduce the nonlinear problem to a sequence of linear boundary value problems. The leading term solution again consists of a rotating disc solution and the solution of a problem in linear bending of a "pre-stressed" flat plate by a (nonuniform) distribution of pressure load.

On the other hand, if $k_G^2 \gg k_V/k_G$, then it would be more efficient to work with a perturbation solution in k_V/k_G. The leading term solution in that case would be a linear bending of a shallow shell of revolution by the combined radial and axial loads which appear on the right side of (3.15) and (3.16).

Region (1.3): $k_G \geq 1$ $(k_V, k_H < 1)$

Let $k_L = \max\,[k_V, k_H]$ and

$$\{\hat{f}, \hat{g}\} = k_L\{f, g\}. \tag{3.18}$$

Then (3.2) and (3.3) may then be written as

$$\frac{1}{k_G} L[f] - \frac{1}{x}\left(s + \frac{k_L}{k_G} f\right) g = -\frac{k_V}{k_L k_G}\frac{1 - x^2}{2x} \tag{3.19}$$

$$\frac{1}{k_G} L[g] + \frac{1}{x}\left(s + \frac{1}{2}\frac{k_L}{k_G} f\right) f = -\frac{k_H}{k_L k_G} x. \tag{3.20}$$

The boundary conditions now consist of (3.10), (3.11a) and

$$g'(x_i) - \frac{\nu}{x_i} g(x_i) + \frac{k_H}{k_L}\frac{x_i^2}{3 + \nu} = 0. \tag{3.21}$$

With $k_L/k_G < 1$ and $\{k_V/k_L, k_H/k_L\} \leq 1$, linearization of the new BVP seems possible by a perturbation solution in k_L/k_G giving us as the leading term solution the conventional linear bending problem for a shallow shell of revolution. This is in fact appropriate for $k_H = 0$ and can be done properly by setting $\hat{f} = k_V f/k_G^2$ and $\hat{g} = k_V g/k_G$ instead. The same is not true in general for $k_H \neq 0$ (see [2,29]) and we should use the scaling for Region III to be discussed later.

(c) *High load intensity range*

Having completed the analysis of the light load magnitude range characterized by $k_V < 1$ and $k_H < 1$, we now turn to the load intensity range $k_V \geq 1$ *or* $k_H \geq 1$. As in the flat plate case, we expect the shell to respond in a qualitatively different way for $k_H^3 < k_V^2$ and for $k_H^3 > k_V^2$. We consider in this section the load magnitude ranges, $\{k_V > k_G^3, k_V^2 > k_H^3\}$ and $\{k_H > k_G^2, k_H^3 > k_V^2\}$. In the load-geometric parameter space $\{k_V, k_H, k_G\}$ with the k_G axis pointing upward (Fig. 3), these two high load magnitude ranges

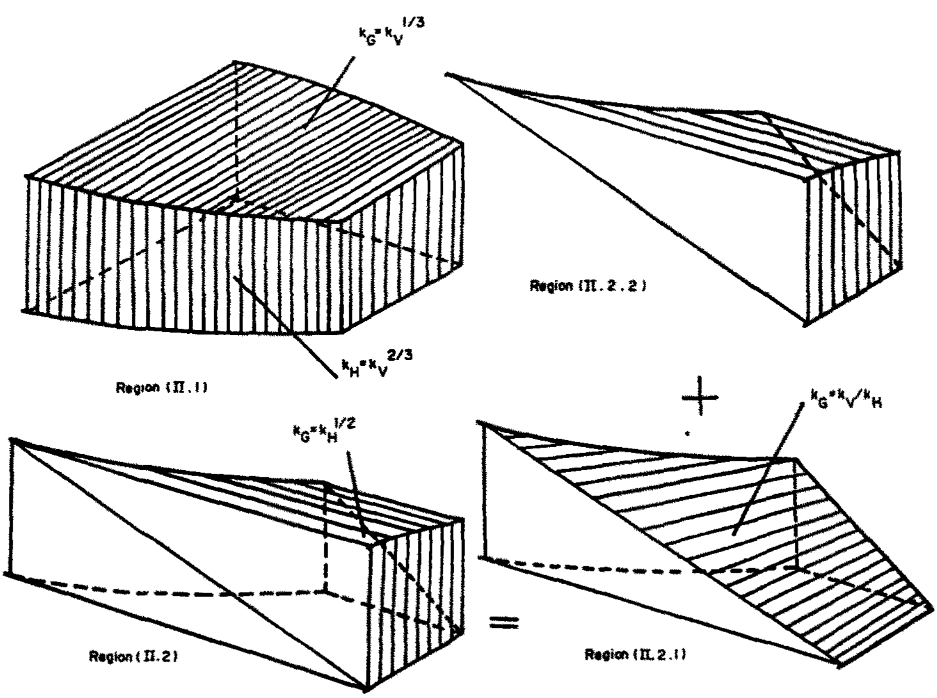

Fig. 5. Region (II) of the parameter space for shells.

span the positive octant of the space outside the unit square column (anchored at the origin) and below the two intersecting surfaces $k_G = k_V^{1/3}$ and $k_G = k_H^{1/2}$. The two subregions $k_H^3 < k_V^2$ and $k_H^3 > k_V^2$ are designated as Region (II.1) and Region (II.2), respectively.

Region (II.1): $k_H^3 < k_V^2$ and $k_G^3 < k_V$.

For $k_G \ll 1$, we expect the shell behavior is platelike; therefore we use the scaling (2.38) (with $\hat{f} = k_V^{1/3} f$ and $\hat{g} = k_V^{2/3} g$) for the plate case and write the two governing differential equations as

$$\frac{1}{k_V^{2/3}} L[f] - \frac{1}{x}\left(\frac{k_G}{k_V^{1/3}} s + f\right) g = -\frac{1 - x^2}{2x} \tag{3.22}$$

$$L[g] + \frac{1}{x}\left(\frac{k_G}{k_V^{1/3}} s + \frac{1}{2} f\right) f = -\frac{x}{\mu_l} \tag{3.23}$$

where $\mu_l^3 = k_V^2/k_H^3$ is as defined in (2.29). The boundary conditions take the form (3.10), (3.11a) and (2.41). It is clear from (3.22) and (3.23) that the same scaling may be used for all $k_G < k_V^{1/3}$, and a perturbation solution in $k_G/k_V^{1/3}$ is appropriate. The leading term solution provides an accurate approximation of the exact solution if $k_G \ll k_V^{1/3}$. This leading term solution is just the solution for the flat plate problem in Region (3) already analyzed. The perturbation solution is not expected to be useful for $k_G^3 \approx k_V$.

Region (II.2): $k_H \geq k_V^{2/3}$ and $k_G < k_H^{1/2}$.

The proper scaling in this region depends on whether we have $k_G < k_V/k_H$ or $k_G > k_V/k_H$. We designate these two subregions as Region (II.2.1) and Region (II.2.2), respectively.

For the subregion (II.2.1) with $k_G k_H < k_V$, we expect the shell to behave nearly as a flat plate, especially if $k_G \ll 1$. Therefore, the scaling (2.26) (with $\hat{f} = (k_V/k_H) f$ and $\hat{g} = k_H g$) for a relatively fast spinning plate should apply for $k_G \ll 1$. With this scaling, (3.2) and (3.3) may be written as

$$\frac{1}{k_H} L[f] - \frac{1}{x}\left(\frac{k_G k_H}{k_V} s + f\right) g = -\frac{1 - x^2}{2x} \tag{3.24}$$

$$L[g] + \frac{1}{x} \mu_l^3 \left(\frac{k_G k_H}{k_V} s + \frac{1}{2} f\right) f = -x \tag{3.25}$$

while the boundary conditions are as given by (3.10) and (3.11). Both $k_G k_H/k_V$ and μ_l^3 are smaller than unity. We may therefore consider a perturbation in $k_G k_H/k_V$. The leading term solution in this range of the parameter values has already been worked out for Region (2) of the plate case.

For the complementary subregion (II.2.2) with $k_G k_H > k_V$, the pressure load is too small to prevent the flattening effect of the radial load. For the shell to be nearly flattened out, we have $\phi + \xi = \bar{\phi}(k_G s + \hat{f}) \cong 0$. Therefore, we take in this region

$$\hat{f} = k_G f, \qquad \hat{g} = k_H g \tag{3.26}$$

and write (3.2) and (3.3) as

$$\frac{1}{k_H} L[f] - \frac{1}{x}(s + f) g = -\frac{k_V}{k_G k_H} \frac{1 - x^2}{2x} \tag{3.27}$$

$$L[g] + \frac{k_G^2}{k_H} \frac{1}{x}\left(s + \frac{1}{2} f\right) f = -x. \tag{3.28}$$

The boundary conditions now take the form (3.10) and (3.11).

With $k_G^2/k_H < 1$, we seek a perturbation solution in k_G^2/k_H to reduce the nonlinear problem to a sequence of linear problems. The leading term for g is the solution of a rotating disc problem while the leading term solution for f is the solution for the linear bending of a "pre-stressed" flat plate by a nonuniform axial load distribution. The perturbation solution is not expected to be useful near $k_G^2 = k_H$.

When $k_H \gg 1$ and $k_G k_H \gg k_V$, we have to a first approximation, $s + f = 0$ except for layer phenomena. The shell is more or less flattened out whenever the radial load is sufficiently high, i.e. when the shell is spinned at a sufficiently high constant angular velocity.

(d) *Moderate load intensity range*

The only range of the load and geometric parameter values not analyzed up to this point is the region in the parameter space outside the unit square column, $0 \le k_H < 1$ and $0 \le k_V < 1$, and above the two intersecting surface $k_G = k_V^{1/3}$ and $k_G = k_H^{1/2}$, i.e. $k_G^3 > k_V$ and $k_G^2 > k_H$. The projection of the intersection curve on the k_H, k_V-plane is the base curve $k_H^3 = k_V^2$. We designate this remaining region as Region III and consider separately the two subregions (III.1) where $k_H^3 < k_V^2$ and (III.2) where $k_H^3 > k_V^2$. Thus, for any $k_G > 0$ plane in the parameter space, we have the light load range $\{k_V \le 1, k_H \le 1\}$, and the high load range $\{k_V \ge k_G^3$ or $k_H \ge k_G^2\}$. The moderate load range occupies the in-between region of the first quadrant of the k_G = constant plane.

Region (III.1): $k_H^3 < k_V^2$ (and $k_G^3 > k_V > 1$).

In this region, we take

$$\hat{f} = k_G f, \qquad \hat{g} = k_V^{2/3} g \tag{3.29}$$

and write (3.2) and (3.3) as

$$\frac{1}{k_V^{2/3}} L[f] - \frac{1}{x}(s + f)\, g = -\left(\frac{k_V}{k_G^3}\right)^{1/3} \frac{1 - x^2}{2x} \tag{3.30}$$

$$\left(\frac{k_V}{k_G^3}\right)^{2/3} L[g] + \frac{1}{x}\left(s + \frac{1}{2} f\right) f = -\frac{k_H}{k_G^2} x. \tag{3.31}$$

Correspondingly, the boundary conditions take the form (3.10), (3.11a) and

$$g'(x_i) - \frac{\nu}{x_i} g(x_i) + \left(\frac{k_H^3}{k_V^2}\right)^{1/3} \frac{x_i^2}{3 + \nu} = 0 \tag{3.32}$$

with $k_H/k_G^2 < (k_V/k_G^3)^{2/3} \le 1$, we seek a perturbation solution in k_H/k_G^2. The leading term solution is identical to the case of no spinning which has already been analyzed in [6–10].

Region (III.2): $k_H^3 > k_V^2$ (and $k_G^2 > k_H > 1$).

The shell is likely to be flattened out in this region. Therefore, we take

$$\hat{f} = k_G f, \qquad \hat{g} = k_G^2 g \quad . \tag{3.33}$$

and write (3.2) and (3.3) as

$$\frac{1}{k_G^2} L[f] - \frac{1}{x}(s + f)\, g = -\frac{k_V}{k_G^3} \frac{1 - x^2}{2x} \tag{3.34}$$

$$L[g] + \frac{1}{x}\left(s + \frac{1}{2} f\right) f = -\frac{k_H}{k_G^2} x. \tag{3.35}$$

Correspondingly, the boundary conditions take the form (3.10), (3.11a) and

$$g'(x_i) - \frac{\nu}{x_i} g(x_i) + \frac{k_H}{k_G^2} \frac{x_i^2}{3 + \nu} = 0. \tag{3.36}$$

With $k_V/k_G^3 < (k_H/k_G^2)^{3/2} \leq 1$ (which follows from $k_V^2 < k_H^3$), a perturbation solution in k_V/k_G^3 is appropriate. The leading term solution is identical to the solution for no pressure load already analyzed in [2].

(e) *Summary of re-scalings and asymptotic solutions*

To summarize the results of this chapter, we write the re-scaling of $\hat{f}$ and $\hat{g}$ in the form

$$\hat{f} = c_f f, \qquad \hat{g} = c_g g \tag{3.37}$$

with the choice of c_f and c_g depending on the value of the parameters k_H, k_V and k_G. The re-scaled form of the two differential equations may be written as

$$c_V L[f] - \frac{1}{x}(c_{Vs}s + c_{fg}f)\,g = -\,c_{Vk}\frac{1 - x^2}{2x} \tag{3.38}$$

$$c_H L[g] + \frac{1}{x}\left(c_{Hs}s + \frac{1}{2}c_{ff}f\right)f = -\,c_{Hk}x. \tag{3.39}$$

Of the four boundary conditions, the form of three of them remains invariant upon re-scaling; the fourth may be written as

$$g'(x_i) - \frac{\nu}{x_i}g(x_i) + c_{HB}\frac{x_i^2}{3 + \nu} = 0. \tag{3.40}$$

The eleven multiplicative re-scaling factors

$$[c_f, c_g], \quad \begin{bmatrix} c_V & c_{Vs} & c_{fg} & c_{Vk} \\ c_H & c_{Hs} & c_{ff} & c_{Hk} \end{bmatrix}, \quad [c_{HB}]$$

completely describe the re-scaled BVP. The re-scaling of the BVP in the various regions of the parameter space will now be summarized in the Table 2 with the help of these multiplicative factors.

The asymptotic solutions in different regions of the parameter space are summarized below. In all cases, the perturbation series involved are not expected to be useful near the boundaries between regions:

Region (I.1): A perturbation solution in $\mu_G < 1$ reduces the problem to solving the corresponding plate problems for Regions (1.1) and (1.2) and a sequence of linear problems which give corrections to the plate solutions.

Region (I.2): A perturbation solution in $k_G < 1$ reduces the problem (in each subregion) to a sequence of linear problems. For the subregion (I.2.1), a perturbation solution in $k_H < 1$ also leads to a similar result.

Region (I.3): Small load amplitudes naturally lead to a perturbation solution in $k_L/k_G < 1$ (where $k_L = \max[k_H, k_V] < 1$) resulting in a linear shell problem (of the singular perturbation type if $k_G \gg 1$) which has been extensively studied in the literatures (e.g. [16, 26]).

Region (II.1): A perturbation solution in $k_G/k_V^{1/3} < 1$ reduces the problem to solving the corresponding flat plate problem and determining corrections to the plate solution from a sequence of linear problems.

Region (II.2.1): A perturbation solution in $k_H k_G/k_V < 1$ reduces the problem to solving the corresponding plate problem and determining corrections to the flat plate solution from a sequence of linear problems. Alternatively, a perturbation solution in

$\mu_l^3 < 1$ has as its leading term the well known rotating disc solution and the solution for the linear bending by a modified axial load of a flat plate pre-stressed by a non-uniform distribution of in-plane (radial) stress resultant induced by the stretching action of the shell. Corrections in the second perturbation solution are also determined by linear problems.

Region (II.2.2): A perturbation solution in $k_G^2 k_H^{-1}$ gives results similar to the perturbation solution in μ_l^3 for Region (II.2.1).

Region (III.1): A perturbation solution in the smaller radial load amplitude factor $k_H/k_G^2 < 1$ has as its leading term the solution for the same problem but without the radial load. This difficult nonlinear BVP has already been analyzed in [6–10].† Corrections to the leading term solution are again determined by a sequence of linear problems.

Region (III.2): A perturbation solution in the smaller amplitude factor $k_V/k_G^3 < 1$ has as its leading term the solution of the same problem for a shell with no pressure loading. This difficult nonlinear BVP has already been analyzed in [2, 27]. Corrections to the leading term solution are again determined by a sequence of linear problems.

4. CONCLUDING REMARKS

In the preceding pages, the problem of a steadily spinning shallow elastic shell of revolution under a uniform pressure distribution has been analyzed and except in narrow ranges of parameter values, solved by perturbation and asymptotic methods. The asymptotic results have been confirmed and complemented by accurate numerical solutions.

For the limiting case of a flat plate, our asymptotic analysis shows that the solution under the combined radial and axial loading may be one of the following four qualitatively different types depending on the particular load combination. For "light" load magnitude, the plate may either behave effectively as a disc pre-stressed by spinning with (secondary) out of plane bending by the uniform pressure, or as a transversely bent plate by the uniform pressure with the deformed slope producing additional (secondary) inplane extension beyond the effect of the steady rotation. At high rotating speed, the plate again behaves effectively as a disc pre-stressed by the centrifugal force and bent out of plane by the uniform pressure, with the (secondary) bending action confined to a narrow layer adjacent to the plate edges. At high pressure loading, the plate behaves effectively as a membrane in finite deformation except for some linear bending action in a boundary layer adjacent to the plate edges. For this case, it can be verified (see [14]) that bending stresses are negligible in the plate interior and are at most of the same order of magnitude as the direct (membrane) stresses near an edge. For design purposes, a solution by the Föppl–Hencky type (or the Reissner type [28]) nonlinear membrane theory should provide a good approximation to the direct stresses and therefore also a good estimate of the bending stress level. Near the borders between load-parameter regions where one of these basic modes of deformations dominates, the expected mode would be modified considerably by the competing neighboring mode(s).

For shallow shells, our asymptotic analysis shows that the structure may exhibit one of three typical modes of deformation, depending on the relative magnitude of the load and geometric parameters. Loosely speaking, the shell is predominantly in a linear membrane or bending mode for sufficiently light load magnitude; "very shallow" shells are nearly plate-like while "very thin" shells at moderate to high load magnitude are usually in a nonlinear membrane or inextensional bending mode. For the first two modes, either bending or stretching action may be dominant in the shell interior, analogous to the flat plate case.

The fact that a shell under the combined radial and axial loading may exhibit one of several modes of behavior is not unexpected. However, our analysis makes precise the ranges of the load and geometrical parameter values for which each of these modes would occur; it also shows how the solution procedure may simplify in each case leading to a complete solution of the problem for the entire parameter space. It is important to note that the same analysis, possibly with some straightforward modifications, applies to shells subject to more general axisymmetric loading conditions and to non-shallow shells. We have focused on a specific class of problems in order to give a unified treatment for these problems important in applications and to put known results for special cases (such as those obtained in references [1, 2, 6–10, 16–23]) in a proper context. Our contributions, however, go well beyond the results for this class of problems; our analysis actually constitutes a general approach to the solution of (small strain) axisymmetric finite deformation problems for shells of revolution under combined axial and radial loading.

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APPENDIX

1. ASYMPTOTIC SOLUTIONS FOR THE FAST SPINNING CASE

From the viewpoint of an asymptotic analysis, the BVP for $k_H \gg 1$ [see (2.27) and (2.28)] is a singular perturbation problem. When $k_H \gg k_V^{2/3}$ (in addition to $k_H \gg 1$), then the leading term perturbation solution in $\mu_i^3 \equiv k_V^2/k_H^3 \ll 1$ for g is determined by (2.31) and (2.32) to be

$$g_0 = A\left[x - \frac{1}{x}\right] + \frac{1}{8}\left[\frac{1}{x} - x^3\right], \quad A = \frac{(1+\nu)[(3+\nu) + (1-\nu)x_i^4]}{8(3+\nu)[(1+\nu) + (1-\nu)x_i^2]}. \tag{A.1}$$

The leading term solution for f is then determined by

$$\begin{gathered} f_0'' + \frac{1}{x}f_0' - \left[\frac{k_H}{x}g_0(x) + \frac{1}{x^2}\right]f_0 = -k_H\frac{1-x^2}{2x} \\ f_0(x_i) = 0, \quad f_0'(1) + \nu f_0(1) = 0. \end{gathered} \tag{A.2}$$

The leading term outer (asymptotic expansion) solution of (A.2) is evidently

$$f_0^{(0)} = \frac{1-x^2}{2g_0} = \frac{4x}{x^2 + 1 - 8A}; \tag{A.3}$$

it does not satisfy either of the boundary conditions for f_0 and hence requires boundary layer corrections.

For the boundary layer correction adjacent to the inner edge, we write the leading term solution which is uniformly valid away from $x = 1$ as

$$f_0 \sim f_0^{(0)}(x) + \bar{f}_0(y), \quad y = \sqrt{k_H}(x - x_i). \tag{A.4}$$

Then $\bar{f}_0$ is determined by the BVP

$$\begin{gathered} \bar{f}_0^{\cdot\cdot} - \alpha^2\bar{f}_0 = 0 \\ \bar{f}_0(0) + \frac{1-x_i^2}{2g_0(x_i)} = 0, \quad \bar{f}_0(y) \to 0 \quad \text{as} \quad y \to \infty \end{gathered} \tag{A.5}$$

where $\alpha^2 = g(x_i)/x_i$. The solution of (A.5) is

$$\bar{f}_0 = -\frac{1-x_i^2}{2g_0(x_i)}e^{-\alpha(\sqrt{k_H})(x-x_i)}. \tag{A.6}$$

Similarly, we write the solution for f_0 uniformly valid away from $x = x_i$ as

$$f_0 \sim f_0^{(0)}(x) + \tilde{f}_0(t), \qquad t = k_H^{1/3}(x - 1). \tag{A.7}$$

With

$$g_0(x) = B(1 - x) + O([1 - x]^2), \qquad B = \frac{1}{2}(1 - 4A) \tag{A.8}$$

near $x = 1$, we find that $\tilde{f}_0(t) = k_H^{1/3}\tilde{f}(t)$ with $\tilde{f}$ determined by the BVP

$$\begin{cases} \tilde{f}^{\cdot\cdot} + Bt\tilde{f} = 0, \\ \tilde{f}^{\cdot}(0) = C_0 \equiv \dfrac{8A - 4}{(1 - 4A)^2}, \qquad \tilde{f}(t) \to 0 \quad \text{as} \quad t \to -\infty. \end{cases} \tag{A.9}$$

The solution of (A.9) is

$$\tilde{f}(t) = C_0\Gamma\left(\frac{1}{3}\right)\left(\frac{3}{B}\right)^{1/3} A_i([k_H B(1 - x)^3]^{1/3}) \tag{A.10}$$

where $A_i(z)$ is the Airy function.

As $\bar{f}_0$ is exponentially small for $x > x_i$ and $\tilde{f}_0$ is exponentially small for $x < 1$, the uniformly valid solution for f_0 in the whole interval $0 \le x \le 1$ is given by

$$\begin{aligned} f_0 &= f_0^{(0)}(x) + \bar{f}_0(y) + k_H^{1/3}\tilde{f}(t) \\ &= \frac{1-x^2}{2g_0(x)} - \frac{1-x_i^2}{2g_0(x_i)}e^{-\alpha(\sqrt{k_H})(x-x_i)} + k_H^{1/3}C_0\Gamma\left(\frac{1}{3}\right)\left(\frac{3}{B}\right)^{1/3}A_i([k_H B(1 - x)^3]^{1/3}). \end{aligned} \tag{A.11}$$

2. ASYMPTOTIC SOLUTION FOR THE HIGH PRESSURE RANGE

Suppose we write the solution of our BVP in Region (3) as

$$\begin{pmatrix} f \\ g \end{pmatrix} \sim \begin{pmatrix} F_0 \\ G_0 \end{pmatrix} + \begin{pmatrix} \bar{f} \\ \bar{g} \end{pmatrix} \tag{A.12}$$

where $G_0(x)$ and $F_0(x)$ are the nonlinear membrane solutions for f and g, respectively, as determined by (2.42). Upon substituting (A.12) into (2.39) and (2.40) and the four relevant boundary conditions and then observing (2.42), we get

$$\epsilon^2\left[\bar{f}'' + \frac{1}{x}\bar{f}' - \frac{1}{x^2}\bar{f}\right] - \frac{1}{x}\bar{f}\bar{g} - \frac{1}{x}G_0(x)\bar{f} - \frac{1}{x}F_0(x)\bar{g} = -\epsilon^2\left[F_0'' + \frac{1}{x}F_0' - \frac{1}{x^2}F_0\right] \tag{A.13}$$

$$\bar{g}'' + \frac{1}{x}\bar{g}' - \frac{1}{x^2}\bar{g} + \frac{1}{2x}\bar{f}^2 + \frac{1}{x}F_0(x)\bar{f} = 0 \tag{A.14}$$

with

$$\begin{gathered} \bar{g}(1) = 0, \quad \bar{f}'(1) + \nu\bar{f}(1) + F_0'(1) + \nu F_0(1) = 0 \\ \bar{g}'(x_i) - \frac{\nu}{x_i}\bar{g}(x_i) = 0, \quad \bar{f}(x_i) + F_0(x_i) = 0 \end{gathered} \tag{A.15}$$

where $\epsilon = k_{\bar{\nu}}^{-1/3}$. The new problem for $\bar{f}$ and $\bar{g}$ is again a singular perturbation problem for $k_\nu^{1/3} \gg 1$. We will not be concerned with the outer solution of this new problem as it is of higher order in ϵ compared to G_0 and F_0.

For an inner solution near x_i, we note that a distinguished limit of the problem (A.13–A.15) as $\epsilon \to 0$ consists of

$$\bar{f} \sim \bar{f}_0(y), \quad \bar{g} \sim \epsilon^2\bar{g}_0(y), \quad y = \frac{x - x_i}{\epsilon} = k_\nu^{1/3}(x - x_i) \tag{A.16}$$

with $\bar{f}_0$ and $\bar{g}_0$ being the solution of the BVP

$$\begin{cases} \bar{f}_0^{\cdot\cdot} - \alpha^2\bar{f}_0 = 0, \quad \bar{g}_0^{\cdot\cdot} + \dfrac{1}{2x_i}\bar{f}_0^2 + \dfrac{1}{x_i}F_0(x_i)\bar{f}_0 = 0, \\ \bar{f}_0(0) + F_0(x_i) = 0, \quad \bar{g}_0^{\cdot}(0) = 0 \\ \bar{g}_0(y), \quad \bar{f}_0(y) \to 0 \quad \text{as} \quad y \to \infty \end{cases} \tag{A.17}$$

where $(\ )^{\cdot} = d(\ )/dy$ and $\alpha^2 = G_0(x_i)/x_i$ (known to be positive from the nonlinear membrane solution), and where the limit of $y \to \infty$ is understood to mean $\epsilon \to 0$, $\eta(\epsilon) \to 0$ $\eta(\epsilon)/\epsilon \to \infty$ (for some suitable $\eta(\epsilon)$) and (the intermediate variable) $z \equiv \epsilon y/\eta(\epsilon) = O(1)$. The solution of this problem is

$$\begin{aligned} \bar{f}_0(y) &= -F_0(x_i)e^{-\alpha y} = -F_0(x_i)e^{-\alpha(x - x_i)/\epsilon} \\ \bar{g}_0(y) &= \frac{3\beta}{4\alpha}\frac{x - x_i}{\epsilon} + C_0 + \frac{\beta}{\alpha^2}\left[e^{-\alpha(x - x_i)/\epsilon} - \frac{1}{8}e^{-2\alpha(x - x_i)/\epsilon}\right] \end{aligned} \tag{A.18}$$

where $\beta = F_0^2(x_i)/x_i$ and where C_0 is a constant to be determined in conjunction with the next order term in the asymptotic expansion (and may be set equal to zero for a leading term solution).

For an inner solution near $x = 1$, we note that a distinguished limit of the BVP in this case consists of

$$\begin{gathered} \bar{f} \sim \sqrt{\epsilon}\hat{f}_0(t) = k_{\bar{\nu}}^{-1/6}\hat{f}_0(t), \quad \bar{g} \sim \epsilon^{3/2}\hat{g}_0(t) = k_{\bar{\nu}}^{-1/2}\hat{g}_0(t) \\ t = \frac{x - 1}{\sqrt{\epsilon}} = k_\nu^{1/6}(x - 1) \end{gathered} \tag{A.19}$$

with $\hat{f}_0$ and $\hat{g}_0$ determined by the BVP

$$\begin{gathered} \hat{f}_0^{\cdot\cdot} - F_0(1)\hat{g}_0 = 0, \quad \hat{g}_0^{\cdot\cdot} + F_0(1)\hat{f}_0 = 0 \\ \hat{g}_0(0) = 0, \quad \hat{f}_0^{\cdot}(0) + F_0'(1) + \nu F_0(1) = 0 \\ \hat{g}_0, \hat{f}_0 \to 0 \quad \text{as} \quad t \to -\infty \end{gathered} \tag{A.20}$$

where $(\ )^{\cdot} \equiv d(\ )/dt$ and where the limit $t \to -\infty$ is understood to mean $\sqrt{\epsilon} \to 0$, $\zeta(\epsilon) \to 0$, $\zeta(\epsilon)/\sqrt{\epsilon} \to \infty$ (for some suitable $\zeta(\epsilon)$) and (the intermediate variable) $s \equiv \sqrt{\epsilon}t/\zeta(\epsilon) = O(1)$. The solution of this problem is

$$\begin{aligned} \hat{f}_0 &= -\frac{\sqrt{2}}{\gamma}[F_0'(1) + \nu F_0(1)]\, e^{\gamma t/\sqrt{2}}\cos(\gamma t/\sqrt{2}). \\ \hat{g}_0 &= \frac{\sqrt{2}}{\gamma}[F_0'(1) + \nu F_0(1)]\, e^{\gamma t/\sqrt{2}}\sin(\gamma t/\sqrt{2}). \end{aligned} \tag{A.21}$$

where $\gamma^2 = F_0(1)$.

With $\{\bar{f}_0, \bar{g}_0\}$ and $\{\tilde{f}_0, \tilde{g}_0\}$ exponentially small away from $x = x_i$ and $x = 1$, respectively, we may write the leading term composite solution as

$$
\begin{aligned}
f &\sim F_0(x) - F_0(x_i)e^{-\alpha(x-x_i)/\epsilon} \\
&\qquad - \frac{\sqrt{2\epsilon}}{\gamma}[F_0'(1) + \nu F_0(1)]e^{\gamma(x-1)/\sqrt{2\epsilon}} \cos(\gamma[x-1]/\sqrt{2\epsilon}) \\
g &\sim G_0(x) + \epsilon^2\left[\frac{3\beta}{4\alpha}\frac{x-x_i}{\epsilon} + \frac{\beta}{\alpha^2}\left(e^{-\alpha(x-x_i)/\epsilon} - \frac{1}{8}e^{-2\alpha(x-x_i)/\epsilon}\right)\right] \\
&\qquad + \frac{\sqrt{2\epsilon^3}}{\gamma}[F_0'(1) + \nu F_0(1)]e^{\gamma(x-1)/\sqrt{2\epsilon}} \sin(\gamma[x-1]/\sqrt{2\epsilon}).
\end{aligned}
\tag{A.22}
$$

For $k_V^{1/6} \gg 1$, the nonlinear membrane solution $F_0(x)$ alone is an adequate approximation for $f(x)$ away from the edge. However, it is clear that $F_0(x)$ is not dominant near the inner edge and the edge zone solution $\bar{f}_0(y)$ must also be included. Furthermore, we have $\tilde{f}_0'(t) = \frac{1}{\sqrt{\epsilon}}O(\tilde{f}_0)$ near $x = 1$, so that the outer solution alone does not give an accurate approximation of the bending stresses of the plate near the outer edge.

In contrast, the nonlinear membrane solution $G_0(x)$ is the dominant part of $g(x)$ throughout the entire plate, the edge zones included, for $k_V^{1/6} \gg 1$. Furthermore, we have $d\bar{g}/dx = O(\epsilon)$ at most for $x_i \le x \le 1$. It follows that the stretching action of the entire plate in region (3) of the k_H, k_V-plane is accurately described by the Föppl–Hencky nonlinear membrane solution alone whenever $k_V^{1/6} \gg 1$.